Lamplighter Group

Based on Office Hour 15 from Office Hours with a Geometric Group Theorist by Matt Clay and Dan Margalit

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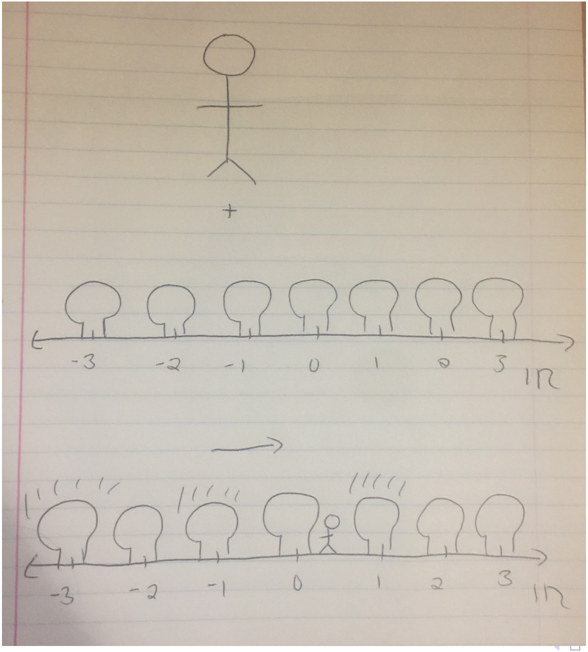
Intuition for the Lamplighter Group



Definition

Question What is the lamplighter group?

Define t as the generator that corresponds to the lamplighter moving one integer value to the right (with t^{-1} moving them one integer value to the left.

Define *a* as the generator that has the lamplighter change the state of the bulb wherever he is currently.

Definition continued

Example

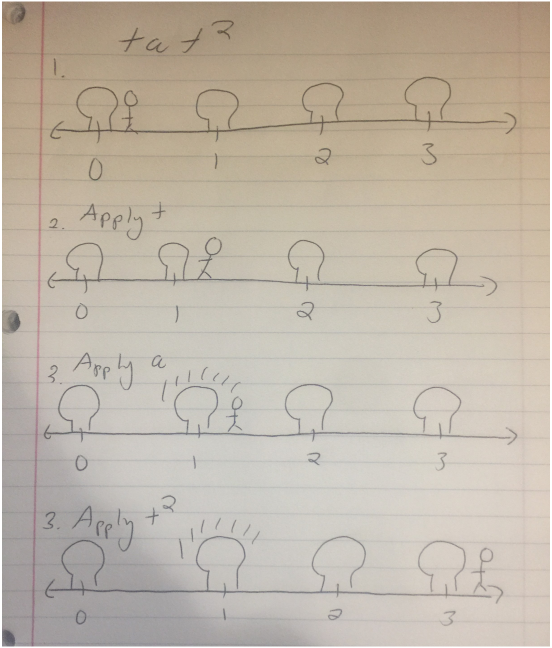
Suppose the lamplighter starts at initial position 0 and no lights are on, then the word t^2atat^{-5} corresponds to lightbulbs being turned on at 2 and 3 with the final position being -2.

With this in mind, the lamplighter group is just set the set of words of t and a with the group multiplication being concatenation of words.

Two words are equal if they induce the same state.

Two states are equal if they have lightbulbs turned on at the same integers and the final position of the lamplighter is the same in both.

Picture of Another Example



Ordered Pair Perspective

We can also define elements as ordered pairs.

Definition

Element of Lamplighter Group
An element of the lamplighter group is an ordered pair where the first entry is a finite set of integers that designate which lightbulbs are turned on and the second entry is the final position of the lamplighter.

Example The ordered pair $(\{-2, 10, 7, -67\}, 9)$ denotes the element that has lightbulbs turned on at $-67, -2, 7, 10$ with the lamplighter's final position at 9.

Comment

Observe that when we give a word in terms of t and a to describe an element of the lamplighter group, we must specify an initial position while the same is not true in the ordered pair definition.

Comment

The amount of lightbulbs that are turned on must be finite, yet can be arbitrary large.

Group Presentation

Generators: t, a as defined earlier.

Relations: Changing the state of any lightbulb twice will always result in the initial state and so $a^2 = 1$.

Conjugates of the form $t^k a t^{-k}, t^l a t^{-l}$ will always commute.

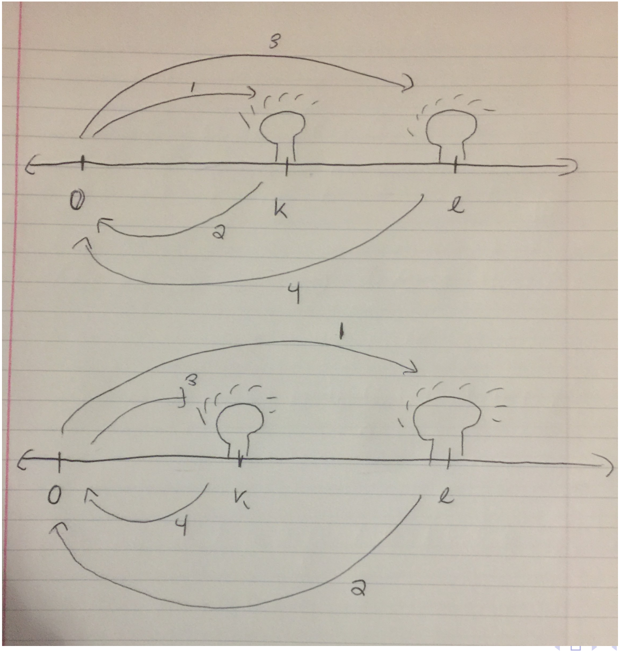
$t^k a t^{-k}$ describes the lamplighter moving k units to the right, changing the state of the bulb, and coming back while $t^l a t^{-l}$ describes the same process for l units to the right.

Therefore we give the following presentation of the lamplighter group.

Presentation

Lamplighter Group $=< t, a | a^2 = 1, [t^k a t^{-k}, t^l a t^{-l}] = 1 >$.

Picture of conjugates commuting



Matrix Perspective

An element of the lamplighter group can be described as an ordered pair (P, k) where $k \in \mathbb{Z}$ and $P = \sum_{i \in Z} c_i x^i$ where $c_i \in \mathbb{Z}/2\mathbb{Z}$. This polynomial encodes which lightbulbs are turned off since the coefficient describes the state of the bulb while the exponent on formal variable x describes the integer where the lightbulb is.

The set of all matrices of the form $\begin{pmatrix} t^k & P \\ 0 & 1 \end{pmatrix}$ where P ranges over all polynomials of x, x^{-1} with coefficients in $\mathbb{Z}/2\mathbb{Z}$ where only finite number are 1 and $k \in \mathbb{Z}$, form a group isomorphic to the lamplighter group.

Matrix Perspective continued

Claim The lamplighter group is isomorphic to the polynomial matrix group

$$\begin{pmatrix} t^k & P \\ 0 & 1 \end{pmatrix}$$

where $k \in \mathbb{Z}$ and P ranges over all polynomials of x, x^{-1} with coefficients in $\mathbb{Z}/2\mathbb{Z}$ where only a finite number are 1.

Explanation We associate t with the matrix $\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$ and associate a with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

By matrix multiplication we see that $a^2 = (\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
We omit the computations but we can do the same verification for the relation of conjugates commuting which can be verified through matrix multiplication.

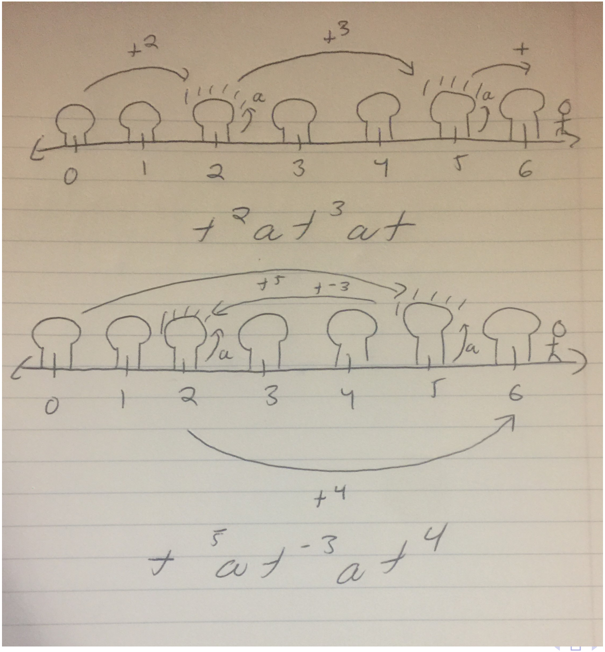
Now let us look at the kernel of this map. To get a matrix of the form $\begin{pmatrix} t^k & P \\ 0 & 1 \end{pmatrix}$ into the identity matrix, we must have that $t^k = 1$ and P is polynomial just the 0 constant. Therefore we must have that $k = 0$ and P has coefficient 0 in each term. This is describing the state of the lamplighter staying at 0 without moving or turning on any lights.

Word Problem

Question How can we tell that two elements of the lamplighter group, here they are words in terms of t and a given some initial position, are equal?

As an example, starting at 0 we see that the words t^2at^3at and $t^5at^{-3}at^4$ are equal because they both have bulbs illuminated at 2 and 5 with the final position of the lamplighter at 6 and thus induce the same state, yet the words themselves look different.

Picture of this Example



Solution to Word Problem

Two Solutions:
1) Brute Force, doable yet boring.
Why is this doable? The number of bulbs is restricted to a finite number and the lamplighter must end at a final position eventually.
2) We turn any word into the form of a product of conjugates and then order them by exponent, with possibly some leading or trailing t or a term. Once we do this we just look and see if the words are equal in each of their letters. As an example, let us do this process for the words t^2at^3at and $t^5at^{-3}at^4$.

Observe that

$$\begin{aligned} t^2at^3at =& t^2at^{-2}t^5at^{-5}t^6 \\ =& (t^2at^{-2})(t^5at^{-5})t^6 \end{aligned}$$

and

$$\begin{aligned} t^5at^{-3}at^4 =& t^5at^{-5}t^2at^{-2}t^6 \\ =& (t^5at^{-5})(t^2at^{-2})t^6 \\ =& (t^2at^{-2})(t^5at^{-5})t^6. \end{aligned}$$

Thus we see that the two words are in fact equal which confirms our expectations that the elements themselves are equal.